

On a planar circular restricted charged three-body problem

Abimael Bengochea • Claudio Vidal

Abstract We introduce a circular restricted charged three-body problem on the plane. In this model, the gravitational and Coulomb forces, due to the primary bodies, act on a test particle; the net force exerted by some primary body on the test particle can be attractive, repulsive or null. The restricted problem is obtained by the general planar charged three-body problem considering one mass of the three bodies going to zero. We obtain necessary restrictions for the parameters that appear in the problem, in order to be well defined. Taking into account such restriction, we study the existence and linear stability of the triangular equilibrium solutions, as well as its location in the configuration space. We also obtain necessary and sufficient conditions for the existence of the collinear equilibrium solutions.

Keywords Hamiltonian Vector Fields, charged three-body problem, circular restricted charged three-body problem.

1 Introduction

The force exerted by electric \mathbf{E} and magnetic field \mathbf{B} on a particle of velocity \mathbf{v} and charge q is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, and it is denominated Lorentz's force (Goldstein 1980; Jackson 1975). The exact description of this problem is formulated in the relativistic context. Nevertheless, when the speed of the particle is much smaller than the one of the light c (around $300,000\text{km/s}$), the dominant term is the Coulombian since the first relativistic

correction is of order $(v/c)^2$, as it is shown by the Lagrangian of Darwin (Jackson 1975). In the study of charged particles, the gravitational interaction is commonly of smaller magnitude than the electromagnetic one, whereas in celestial problems it happens the opposite.

Several works have been written about restricted models concerning three point particles (it is assumed that one of the bodies does not affect the motion of the other two particles) with additional interactions to the gravitational one, for instance the Coulomb or photogravitational cases (the gravitational force is attractive, the Coulomb can be either attractive or repulsive, and the photogravitational only repulsive, in a generic way - the zero value is allowed in all of them).

Radzievskii (1950, 1953) introduced the photogravitational restricted three-body problem, and Dionysiou and Stamou (1989) a restricted charged three-body problem. Schuerman (1980) studied the stability, and location on the configuration space, of the equilibrium solutions when radiation pressure and Poynting-Roberts forces are included. In the photogravitational case, Kunitsyn and Tureshbaev (1983) studied the existence and stability of the collinear equilibrium solutions, Lukyanov (1984) the existence of the triangular equilibrium solutions, and the collinear equilibrium solution, in a parametric way. In the same model, Kunitsyn and Tureshbaev (1985a) considered the existence of the collinear equilibrium solutions, and Kunitsyn and Tureshbaev (1985b) the existence and stability of the equilibrium triangular solutions. Simmons (1985) studied the restricted case with radiation pressure; part of his study is concerned with the analysis of the triangular, collinear and spatial equilibrium solutions. In the photogravitational restricted problem, Lukyanov (1986) studied the stability of the collinear and triangular equilibrium solutions, as well as its location in the configuration space.

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Some of the studies present the same potential function, including our own. However, the parameters, as well as its allowed values, can be different. For instance, in the restricted charged three-body problem, Dionysiou and Vaiopoulos (1987) consider a different parameterization from the one used in this work. In fact, the potential function is $V = \frac{q-\mu}{\rho_1} + \frac{\mu-q}{\rho_2}$, where $\mu \in (0, 1/2]$, $q \in \mathbb{R}$, and ρ_i denotes the distance between the body $i = 1, 2$ and the third body. On the other hand, in the photogravitational case, Radzievskii (1950), Kunitsyn and Tureshbaev (1983), Lukyanov (1984) and Simmons (1985), consider a parameterization similar to ours. The potential function is $V = \frac{\gamma_1(1-\mu)}{\rho_1} + \frac{\gamma_2\mu}{\rho_2}$, where μ , γ_1 and γ_2 are real parameters, and ρ_i , $i = 1, 2$, the distance between the bodies, as in the charged case. The parameter μ belongs to $(0, 1/2]$, whereas $\gamma_i \in (-\infty, 1]$, $i = 1, 2$. The qualitative properties of the gravitational and photogravitational forces acting on the test body, by the primaries $i = 1, 2$, are defined by the parameter γ_i : if $\gamma_i < 0$ the gravitational force dominates over the photogravitational one, for $\gamma_i = 0$ both forces are equal to each other in magnitude, for $0 < \gamma_i < 1$ the gravitational is the strongest one, and for $\gamma_i = 1$ the photogravitational force is zero. The case $\gamma_i > 1$ has not physical sense because it corresponds to an attractive photogravitational force; usually it is included in the study of the mathematical model. The different selection of units and generality with which these problems have been studied can be appreciated in the number of parameters that appear, that is, two and three respectively.

This paper is organized as follows. In Section 2 we give a brief review of the problem of two charged bodies. Here we emphasize that the necessary condition on the parameters of mass m_i and charge q_i , $i = 1, 2$, for the existence of circular orbits of the particles 1 and 2, is $G - k \frac{q_1 q_2}{m_1 m_2} > 0$ (G is the constant of universal gravitation and k is the constant of Coulomb); this limits the values of the parameters that appear in the restricted problem of three charged bodies on the plane, which we enunciate in Section 3. With this aim, we make the mass of the third body tends to zero, which gives us a well defined restricted circular problem. The potential function associated to this problem is given by $V = \frac{\beta_1(1-\mu)}{\rho_1} + \frac{\beta_2\mu}{\rho_2}$, with parameters μ , β_1 , β_2 , and the usual distances ρ_1 , ρ_2 . Here $\mu \in (0, 1/2]$ and $\beta_i \in \mathbb{R}$ for $i = 1, 2$, with the restriction $(\beta_1 - 1)(\beta_2 - 1) < 1$, which is the necessary condition for having circular solutions for the bodies 1 and 2. Such condition has not been considered in previous studies, for instance Dionysiou and Vaiopoulos (1987), Dionysiou and Stamou (1989), Kunitsyn and Tureshbaev (1983), Kunitsyn and Tureshbaev (1985b), Lukyanov

(1984), Lukyanov (1986), Simmons (1985), reason why our problem acquires a great difference to others already treated and justifies its study. We remark that, if the parameters that appear in the potential are not properly considered, the interaction between the primary bodies will be repulsive and might prevent those bodies from having a circular movement. As in the photogravitational model, the parameters β_i , $i = 1, 2$ determine the qualitative features of the gravitational and Coulomb forces. Actually, we have a similar description for both photogravitational and charged problems, with the exception that $\beta_i > 1$ is physically possible and corresponds to a Coulomb attractive force. Once established the restricted charged three-body problem, in Section 4 we study the existence of triangular and collinear equilibrium solutions. In Section 5 we deal with the linear stability of the triangular equilibrium solutions and its location in the configuration and parameters space. We finish the article with the conclusions of this work.

2 Dynamics of the two-charged problem

Consider an inertial frame of reference. The Hamiltonian associated to bodies 1 and 2 of charges q_1 and q_2 , and masses m_1 and m_2 respectively, with gravitational and Coulombian interaction, is

$$H(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) = \frac{\|\mathbf{p}_1\|^2}{2m_1} + \frac{\|\mathbf{p}_2\|^2}{2m_2} + \frac{kq_1q_2 - Gm_1m_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|}, \quad (1)$$

where $k > 0$ is the Coulomb constant, $G > 0$ is the universal gravitational constant, and $\mathbf{r}_i \in \mathbb{R}^3$, $\mathbf{p}_i \in \mathbb{R}^3$, $i = 1, 2$ are the positions and moments of the bodies with the same index, respectively.

Given the Hamiltonian (1), and the initial conditions, the equations of Hamilton $\dot{\mathbf{r}}_i = H_{\mathbf{p}_i}$, $\dot{\mathbf{p}}_i = -H_{\mathbf{r}_i}$ determine the motion of each body. The equations of motion correspond to a system of differential equations of second order:

$$\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= -(Gm_1m_2 - kq_1q_2) \frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3}, \\ m_2 \ddot{\mathbf{r}}_2 &= -(Gm_1m_2 - kq_1q_2) \frac{\mathbf{r}_2 - \mathbf{r}_1}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3}, \end{aligned}$$

where the dot denotes derivative with respect to time t . In order to classify the solutions, we define $C = Gm_1m_2 - kq_1q_2$. Three cases are identified:

- $C > 0$,
- $C = 0$,
- $C < 0$.

The first one is equivalent to the Kepler's problem whose dynamics is well known. Notice that the condition $C > 0$ is satisfied whenever the charges have different signs, and if the charges have equal signs it is required that $kq_1q_2 < Gm_1m_2$. The case $C = 0$ corresponds to two free particles. The case $C < 0$ is rarely discussed in the literature, therefore we give a brief description of it, following the same steps as in the Kepler's problem (Goldstein 1980). Notice that the system associated to $C < 0$ has the Kepler's constants of motion, namely energy, angular momentum and those related to the center of mass of the system. Therefore, in a generic way, the movement of the two bodies occurs in a fixed plane, which we assume from here on. As first step, we make a symplectic transformation of coordinates, that change the vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2$ to the ones

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1, \quad \mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \\ \mathbf{p} &= \frac{m_1}{m_1 + m_2}\mathbf{p}_2 - \frac{m_2}{m_1 + m_2}\mathbf{p}_1, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2. \end{aligned}$$

The upper vectors correspond to the center of mass of the system, whereas the lowercase ones describe the motion relative to the center of mass of the system. From the previous change of variables it is obtained

$$\begin{aligned} \mathbf{r}_1 &= -\frac{m_2}{m_1 + m_2}\mathbf{r} + \mathbf{R}, \quad \mathbf{r}_2 = \frac{m_1}{m_1 + m_2}\mathbf{r} + \mathbf{R}, \\ \mathbf{p}_1 &= -\mathbf{p} + \frac{m_1}{m_1 + m_2}\mathbf{P}, \quad \mathbf{p}_2 = \mathbf{p} + \frac{m_2}{m_1 + m_2}\mathbf{P}. \end{aligned}$$

By replacing the new variables and $-C = |C|$ in the Hamiltonian (1) we obtain

$$H^*(\mathbf{r}, \mathbf{R}, \mathbf{p}, \mathbf{P}) = \frac{\|\mathbf{p}\|^2}{2\mu} + \frac{\|\mathbf{P}\|^2}{2M} + \frac{|C|}{\|\mathbf{r}\|}, \quad (2)$$

where $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass, and $M = m_1 + m_2$ is the total mass of the system. The Hamiltonian (2) is separable. For the motion relative to the center of mass we have

$$H_{rel}^*(\mathbf{r}, \mathbf{p}) = \frac{\|\mathbf{p}\|^2}{2\mu} + \frac{|C|}{\|\mathbf{r}\|}. \quad (3)$$

It is not difficult to identify the Hill's region for (3). This consists of all the points \mathbf{r} on the configuration space such that

$$h_{rel}^* \geq \frac{|C|}{\|\mathbf{r}\|},$$

where $h_{rel}^* > 0$ is the constant of motion associated to (3). Thus, the points that define this region must satisfy

$$\|\mathbf{r}\| \geq r_0,$$

with $r_0 = \frac{|C|}{h_{rel}^*}$. The Hill's region is outlined in Figure 1.

EDITOR: PLACE FIGURE 1 HERE.

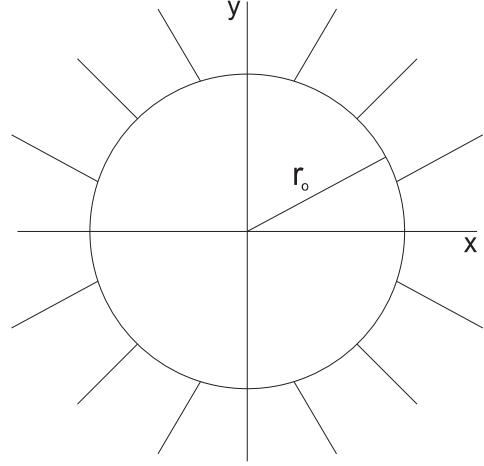


Fig. 1 Hill's region (hatched part) of the two-charged problem, with $h_{rel}^* > 0$, $C < 0$.

Now we introduce polar coordinates by means of using the generating function $S_2(p_x, p_y, \rho, \theta) = p_x\rho \cos(\theta) + p_y\rho \sin(\theta)$:

$$x = \frac{\partial S_2}{\partial p_x} = \rho \cos(\theta),$$

$$y = \frac{\partial S_2}{\partial p_y} = \rho \sin(\theta),$$

$$p_r = \frac{\partial S_2}{\partial \rho} = p_x \cos(\theta) + p_y \sin(\theta),$$

$$p_\theta = \frac{\partial S_2}{\partial \theta} = -p_x \rho \sin(\theta) + p_y \rho \cos(\theta).$$

With this transformation we obtain the new Hamiltonian

$$K(\rho, \theta, p_r, p_\theta) = \frac{1}{2\mu} \left(p_r^2 + \frac{p_\theta^2}{\rho^2} \right) + \frac{|C|}{\rho}, \quad (4)$$

associated to the constant of movement $k^* = h_{rel}^* > 0$. In this Hamiltonian does not appear θ , reason why the canonical momentum $p_\theta = l$ is constant, which corresponds to the conservation of the angular momentum. Replacing $p_\theta = l$ in (4), and considering the reduced energy k^* , it is obtained

$$k^* = \frac{p_r^2}{2\mu} + \frac{l^2}{2\mu\rho^2} + \frac{|C|}{\rho}, \quad (5)$$

where $\frac{l^2}{2\mu\rho^2} + \frac{|C|}{\rho}$ is defined as the effective potential $V_{eff}(\rho)$. The constant of movement (5) is useful to obtain the phase portrait; from this we get $p_r = \pm \sqrt{2\mu(k^* - V_{eff}(\rho))}$. When ρ takes the value $\rho_* = \frac{|C|}{2k^*} (1 + \sqrt{1 + \frac{2l^2k^*}{C^2\mu}})$ the canonical momentum p_r is zero. Then, as k^* grows ρ_* tends to zero, and if $\rho \rightarrow \infty$

then $p_r \rightarrow \pm\sqrt{2\mu k^*}$. The phase portrait is shown in Figure 2.

EDITOR: PLACE FIGURE 2 HERE.

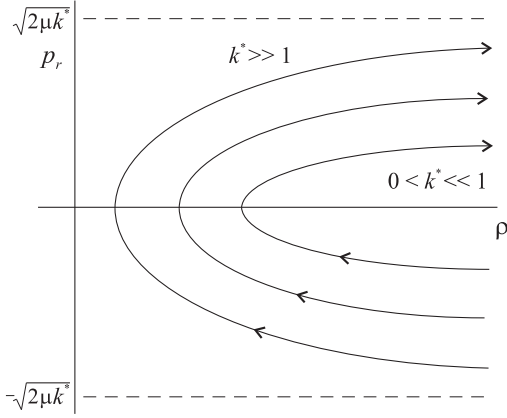


Fig. 2 Phase portrait of the two-charged problem, with $k^* > 0, C < 0$.

The ordinary differential equations obtained from the Hamiltonian (4) can be integrated, as in the Kepler's problem. Defining $u = \rho^{-1}$, and using both constants of motion k^* and l , we have

$$d\theta = -\frac{du}{\sqrt{\frac{2\mu k^*}{l^2} - \frac{2\mu|C|u}{l^2} - u^2}},$$

thus

$$\theta - \theta' = -\arccos\left(\frac{\frac{2\mu|C|}{l^2} + 2u}{\sqrt{\frac{4\mu^2 C^2}{l^2} + \frac{8\mu k^*}{l^2}}}\right),$$

where θ' is a constant of integration. Therefore, ρ and θ are related through the expression

$$\frac{1}{\rho} = c [-1 + e \cos(\theta - \theta')], \quad (6)$$

where

$$c = \frac{\mu|C|}{l^2}, \quad e = \sqrt{1 + \frac{2l^2 k^*}{\mu C^2}}.$$

The equation (6) defines a hyperbola of eccentricity e , whose focus is located at the origin. The hyperbola is sketched in Figure 3; by simplicity we have chosen $\theta' = 0$. Since $e > 1$ we have that $1 > \frac{1}{e} = \cos(\theta_e) > 0$ for some $\theta_e \in (0, \frac{\pi}{2})$, thus the asymptotes of the hyperbola are associated to the angles $\theta_e, 2\pi - \theta_e$.

EDITOR: PLACE FIGURE 3 HERE.

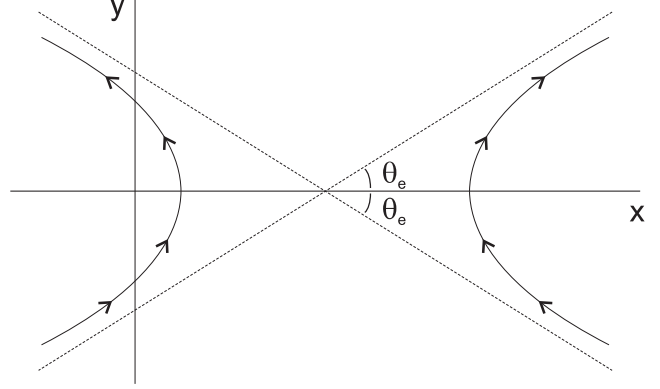


Fig. 3 Behavior of the solutions of the two-charged problem with $k^* > 0, C < 0$.

3 The restricted charged three-body problem

In this Section we introduce the restricted charged three-body problem, as a limiting case of the general charged three-body problem. This restricted problem will appear when the mass of the third body tends to zero.

We remember that the Hamiltonian corresponding to the system of bodies $i = 1, 2, 3$ of charge q_i and mass m_i , in an inertial frame of reference, that interact under the gravitational and Coulomb forces on the plane, is

$$H(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) =$$

$$\sum_{i=1}^3 \frac{\|\mathbf{p}_i\|^2}{2m_i} - \sum_{1 \leq i < j \leq 3} \frac{\lambda_{ij}}{\|\mathbf{r}_i - \mathbf{r}_j\|},$$

where

$$\lambda_{ij} = Gm_i m_j - kq_i q_j.$$

As before, k is the Coulomb constant, G is the universal gravitational constant, and $\mathbf{r}_i \in \mathbb{R}^2$, $\mathbf{p}_i \in \mathbb{R}^2$, $i = 1, 2, 3$ are the positions and momentum of the bodies 1, 2 and 3, respectively.

We introduce the real parameters α_i and C_{ij} by means of

$$q_i = \alpha_i m_i, \quad i = 1, 2, 3,$$

$$C_{ij} = G - k\alpha_i \alpha_j, \quad i, j = 1, 2, 3,$$

with the restriction $\alpha_3 \neq 0$. Notice that this last condition guarantees the existence of charge of the third particle, therefore a different problem from the classic one.

The differential equations associated to the problem of three charged bodies are given by

$$\begin{aligned}\ddot{\mathbf{r}}_1 &= -\frac{\lambda_{12}(\mathbf{r}_1 - \mathbf{r}_2)}{m_1 \|\mathbf{r}_1 - \mathbf{r}_2\|^3} - \frac{\lambda_{13}(\mathbf{r}_1 - \mathbf{r}_3)}{m_1 \|\mathbf{r}_1 - \mathbf{r}_3\|^3}, \\ \ddot{\mathbf{r}}_2 &= -\frac{\lambda_{21}(\mathbf{r}_2 - \mathbf{r}_1)}{m_2 \|\mathbf{r}_2 - \mathbf{r}_1\|^3} - \frac{\lambda_{23}(\mathbf{r}_2 - \mathbf{r}_3)}{m_2 \|\mathbf{r}_2 - \mathbf{r}_3\|^3}, \\ \ddot{\mathbf{r}}_3 &= -\frac{\lambda_{13}(\mathbf{r}_3 - \mathbf{r}_1)}{m_3 \|\mathbf{r}_3 - \mathbf{r}_1\|^3} - \frac{\lambda_{23}(\mathbf{r}_3 - \mathbf{r}_2)}{m_3 \|\mathbf{r}_3 - \mathbf{r}_2\|^3}.\end{aligned}\quad (7)$$

Taking $m_3 \rightarrow 0$ we have that $q_3 \rightarrow 0$, then $\lambda_{13} \rightarrow 0$ and $\lambda_{23} \rightarrow 0$, therefore, by (7), the position vectors \mathbf{r}_1 and \mathbf{r}_2 obey to one problem of two charged bodies, that is

$$\begin{aligned}\ddot{\mathbf{r}}_1 &= -\frac{C_{12}m_2(\mathbf{r}_1 - \mathbf{r}_2)}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3}, \\ \ddot{\mathbf{r}}_2 &= -\frac{C_{12}m_1(\mathbf{r}_2 - \mathbf{r}_1)}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3}.\end{aligned}$$

On the other hand, for the third position vector we have

$$\ddot{\mathbf{r}}_3 = -\frac{C_{13}m_1(\mathbf{r}_3 - \mathbf{r}_1)}{\|\mathbf{r}_3 - \mathbf{r}_1\|^3} - \frac{C_{23}m_2(\mathbf{r}_3 - \mathbf{r}_2)}{\|\mathbf{r}_3 - \mathbf{r}_2\|^3}.$$

The dynamics of the third particle depends on the motion previously chosen for the bodies 1 and 2, and on the parameters

$$C_{13} = G - k\alpha_1\alpha_3, \quad C_{23} = G - k\alpha_2\alpha_3.$$

As we saw in the previous Section, in order to have circular orbits for the primaries, the parameters must satisfy

$$C_{12} = G - k\alpha_1\alpha_2 > 0. \quad (8)$$

Remark 3.1 *Without the restriction (8) on the parameters α_1 and α_2 , or m_1 , m_2 , q_1 and q_2 , the primaries cannot follow circular trajectories, thus the circular restricted charged three-body problem will not be well defined.*

We take a circular solution for the bodies 1 and 2 (inertial frame), where both particles move around their center of mass with constant angular frequency ω . We set $\mathbf{r}_1(t) = c_1(-\cos\omega t, -\sin\omega t)$ as solution for the first body, and $\mathbf{r}_2(t) = c_2(\cos\omega t, \sin\omega t)$ for the second one; c_1 and c_2 are positive constants that satisfy the condition $m_1c_1 = m_2c_2$, which corresponds to define the origin of the system at the center of mass of the bodies 1 and 2. Since the bodies 1 and 2 have a circular trajectory, it is appropriate to take a rotating system whose frequency is ω . The position vectors in the inertial and rotating frames are related by means of the rotation

$$R(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$$

as follows:

$$\mathbf{r}_1(t) = R(t)\mathbf{s}_1, \quad \mathbf{r}_2(t) = R(t)\mathbf{s}_2, \quad \mathbf{r}_3 = R(t)\mathbf{r}.$$

It is not difficult to see that \mathbf{s}_1 and \mathbf{s}_2 lie on the horizontal axis (rotating frame), that is $\mathbf{s}_1 = c_1(-1, 0)$, $\mathbf{s}_2 = c_2(1, 0)$. On the other hand, the equation that describes the movement of the third body in the synodic frame is given by

$$\ddot{\mathbf{r}} = \omega^2\mathbf{r} + 2\omega J\dot{\mathbf{r}} - \frac{C_{13}m_1(\mathbf{r} - \mathbf{s}_1)}{\|\mathbf{r} - \mathbf{s}_1\|^3} - \frac{C_{23}m_2(\mathbf{r} - \mathbf{s}_2)}{\|\mathbf{r} - \mathbf{s}_2\|^3}, \quad (9)$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The second order differential equation (9) can be written in terms of a pair of first order vector differential equations in \mathbb{R}^2 , that is

$$\dot{\mathbf{r}} = \mathbf{v},$$

$$\dot{\mathbf{v}} = 2\omega J\mathbf{v} + \frac{\partial\Omega}{\partial\mathbf{r}},$$

where the function Ω is given by

$$\Omega = \frac{1}{2}\omega^2\mathbf{r}^2 + \frac{C_{13}m_1}{\rho_1} + \frac{C_{23}m_2}{\rho_2}, \quad (10)$$

with $\rho_1 = \|\mathbf{r} - \mathbf{s}_1\|$ and $\rho_2 = \|\mathbf{r} - \mathbf{s}_2\|$.

It is convenient to introduce the parameters

$$\mu_1 = Gm_1, \quad \mu_2 = Gm_2, \quad \alpha_j\sqrt{\frac{k}{G}} = \widetilde{\alpha}_j, \quad j = 1, 2, 3,$$

together with

$$\delta = \text{sgn}(\alpha_3), \quad \beta_j = 1 - \widetilde{\alpha}_j\widetilde{\alpha}_3, \quad j = 1, 2.$$

Notice that four independent physical magnitudes exist: time, distance, mass and charge. We choose ω as unitary frequency (time), the separation between 1 and 2 as unitary distance. Also units of charge and mass such that

$$|\widetilde{\alpha}_3| = 1, \quad \mu_1 + \mu_2 = 1.$$

According to the selection of units of mass, and by the symmetry of the problem, it is enough to consider only one mass parameter belonging to $(0, 1/2]$. For this we use μ , where $0 < \mu \leq 1/2$, $\mu_2 = \mu$ and $\mu_1 = 1 - \mu$. With this we have

$$C_{13}m_1 = (1 - \mu)(1 - \widetilde{\alpha}_1\delta) = (1 - \mu)\beta_1,$$

$$C_{23}m_2 = \mu(1 - \widetilde{\alpha}_2\delta) = \mu\beta_2.$$

Using the condition $(1 - \mu)c_1 = \mu c_2$ (center of mass), the relation $c_1 + c_2 = 1$ (units of distance), and $Gm_1 = 1 - \mu$, $Gm_2 = \mu$ (units of mass), it is obtained that $\mathbf{s}_1 = (-\mu, 0)$ and $\mathbf{s}_2 = (1 - \mu, 0)$. The positions of the three bodies, in the synodic frame, are shown in Figure 4.

EDITOR: PLACE FIGURE 4 HERE.

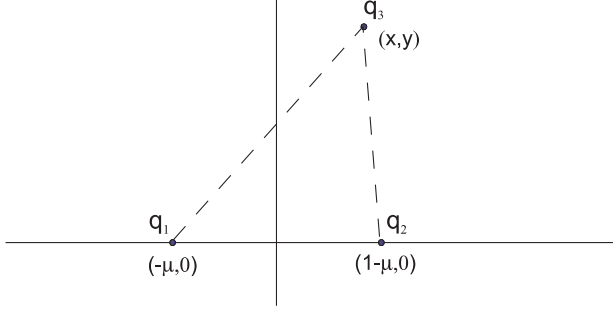


Fig. 4 The restricted circular charged three-body problem in a rotating frame with convenient units.

The parameters that appear in (10) have been written in terms of the new parameters β_1 , β_2 , but still without the restriction associated to C_{12} . For this units, the condition $C_{12} > 0$ is equivalent to $\tilde{\alpha}_1 \tilde{\alpha}_2 < 1$, which implies

$$(\beta_1 - 1)(\beta_2 - 1) < 1. \quad (11)$$

The region defined by (11) is shown in Figure 5.

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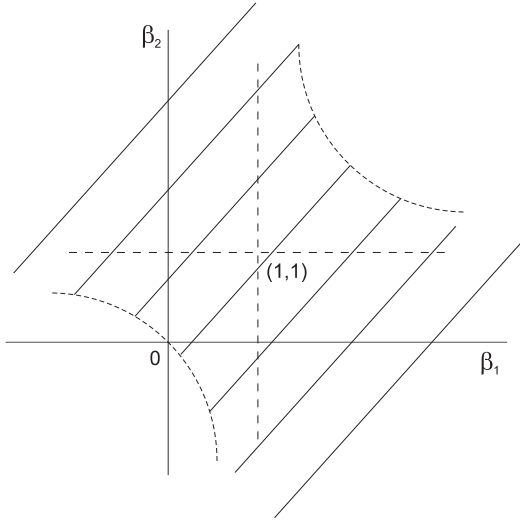


Fig. 5 The hatched part, delimited by $\beta_2 = \frac{1}{\beta_1 - 1} + 1$, defines the allowed values of the parameters β_1 and β_2 .

Remark 3.2 The difference between the circular restricted charged three-body problem and the circular

restricted three-body problem of *Celestial Mechanics* (which is given by $(\beta_1, \beta_2) = (1, 1)$) is in the values that can assume β_1 and β_2 .

In the following, it is described the relation between the forces that the particle $i = 1, 2$, exerts on the third particle, according to the values that β_i can assume:

- $\beta_i < 0$. The Coulomb force is repulsive and predominates over the gravitational one.
- $\beta_i = 0$. The gravitational and Coulomb forces have equal magnitude, being the second one repulsive.
- $0 < \beta_i < 1$. The Coulomb force is repulsive and the gravitational force predominates.
- $\beta_i = 1$. The Coulomb force is null.
- $1 < \beta_i$. The Coulomb force is attractive.

Denoting the coordinates of \mathbf{r} by (x, y) , and taking $\omega = 1$, we have that (9) is equivalent to

$$\ddot{x} - 2\dot{y} - x = V_x, \quad (12)$$

$$\ddot{y} + 2\dot{x} - y = V_y,$$

where

$$V = \frac{\beta_1(1 - \mu)}{\rho_1} + \frac{\beta_2\mu}{\rho_2}, \quad (13)$$

with

$$\rho_1 = \sqrt{(x + \mu)^2 + y^2}, \quad \rho_2 = \sqrt{(x - 1 + \mu)^2 + y^2}.$$

Considering the coordinates

$$p_x = \dot{x} - y, \quad p_y = \dot{y} + x,$$

we obtain that (12) is equivalent to the system

$$\dot{x} = y + p_x = H_{p_x}, \quad \dot{p}_x = V_x + p_y = -H_x, \quad (14)$$

$$\dot{y} = -x + p_y = H_{p_y}, \quad \dot{p}_y = V_y - p_x = -H_y,$$

whose Hamiltonian function is

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + (yp_x - xp_y) - V. \quad (15)$$

The Hamiltonian function (15), the potential function (13), and the restriction (11), define the restricted charged three-body problem.

Remark 3.3 *Dionysiou and Vaiopoulos (1987) considered the circular restricted three-charged-body problem with a different parameterization from the one used in this work. On the other hand, Radzievskii (1950, 1953), Kunitsyn and Tureshbaev (1983), Lukyanov (1984), Simmons (1985), studied the restricted photogravitational problem making use of a parameterization similar to the one we have used in our work. In the photogravitational case it is considered, in addition to the*

gravitational force, a repulsive (or null) force of magnitude inversely proportional to the square of the distance, equivalent to the restriction $\beta_i \in (-\infty, 1]$.

A relevant consequence of (11) is that the equilibrium solutions are confined to stay in a certain region of the configuration space, as we shall see in the following Section.

4 Equilibrium solutions

According to (14), the equilibrium solutions of the circular restricted charged three-body problem must satisfy

$$V_x + x = 0, \quad V_y + y = 0,$$

which implies

$$xf - \mu(1 - \mu) \left(\frac{\beta_1}{\rho_1^3} - \frac{\beta_2}{\rho_2^3} \right) = 0,$$

$$yf = 0,$$

where

$$f = 1 - \frac{\beta_1(1 - \mu)}{\rho_1^3} - \frac{\beta_2\mu}{\rho_2^3}.$$

We can separate the solutions in two types: $y \neq 0$ (non collinear or triangular) and $y = 0$ (collinear). Considering the first case, $y \neq 0$, it is required that $f = 0$, which implies that

$$1 - \frac{\beta_1(1 - \mu)}{\rho_1^3} - \frac{\beta_2\mu}{\rho_2^3} = 0 \quad (16)$$

and

$$\frac{\beta_1}{\rho_1^3} - \frac{\beta_2}{\rho_2^3} = 0 \quad (17)$$

hold. The equation (17) requires that β_1 and β_2 have the same sign. If this happens, the sign must be positive, so that (16) has solution. The corresponding solution is $\rho_1 = \beta_1^{1/3}$, $\rho_2 = \beta_2^{1/3}$. Therefore, it is required that the β -parameters be positive, and satisfy the triangular inequalities $\rho_1 + 1 > \rho_2$, $\rho_2 + 1 > \rho_1$, $\rho_1 + \rho_2 > 1$. Besides this, the restriction (11) must be fulfilled. In order to describe the triangular solutions on the parameters space, we introduce $(\delta_1, \delta_2) = (\beta_1^{1/3}, \beta_2^{1/3})$; we will use them interchangeably. In Figures 6 and 7 it is shown the allowed region for the existence of triangular equilibrium solutions, in the parameters and configuration space, respectively.

EDITOR: PLACE FIGURE 6 HERE.

EDITOR: PLACE FIGURE 7 HERE.

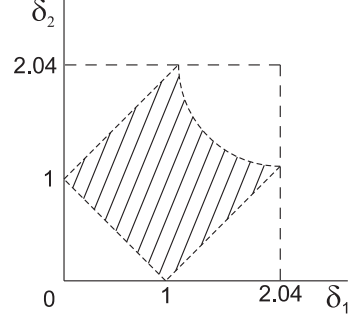


Fig. 6 Allowed region, on the parameters space $(\delta_1, \delta_2) = (\beta_1^{1/3}, \beta_2^{1/3})$, for the existence of triangular equilibrium solutions. The region is delimited by the functions $\delta_2 = \delta_1 + 1$, $\delta_2 = \delta_1 - 1$, $\delta_2^3 = \frac{1}{\delta_1^3 - 1} + 1$.

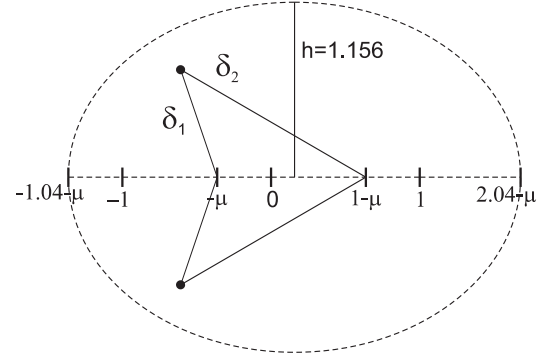


Fig. 7 Allowed region, on the configuration space, for the existence of triangular equilibrium solutions.

There are two equilibrium solutions of this type, one with $y > 0$, and another with $y < 0$. It is not difficult to see that the coordinates of these solutions are $(x, y) = (x_L, \pm y_L)$, where

$$x_L = -\mu + \frac{1}{2}(\beta_1^{2/3} - \beta_2^{2/3} + 1),$$

$$y_L = \frac{1}{2}\sqrt{2(\beta_1^{2/3} + \beta_2^{2/3}) - (\beta_1^{2/3} - \beta_2^{2/3})^2 - 1}.$$

Thus, we have proved that:

Theorem 4.1 *There are two triangular equilibrium solutions in the circular restricted charged three-body problem, whenever β_1 and β_2 belong to the region shown in Figure 6. They are given by*

$$L_4 = (x_L, y_L), \quad L_5 = (x_L, -y_L).$$

Remark 4.1 *This result was proved in the corresponding models by Kunitsyn and Tureshbaev (1985b), Lukyanov (1984, 1986) and Simmons (1985).*

Proposition 4.1 *The parameters β_1 and β_2 set the location of the triangular equilibrium solutions, in the following way:*

- $\beta_2^{1/3} > \sqrt{1 + \beta_1^{2/3}}$. The equilibrium solution is located to the left of the body 1 (see Figure 8).
- $\beta_2^{1/3} = \sqrt{1 + \beta_1^{2/3}}$. The equilibrium solution is above or below the body 1.
- $\beta_1^{1/3} < \sqrt{1 + \beta_2^{2/3}}$ or $\beta_2^{1/3} < \sqrt{1 + \beta_1^{2/3}}$. The equilibrium solution is located between the bodies 1 and 2 (see Figure 9).
- $\beta_1^{1/3} = \sqrt{1 + \beta_2^{2/3}}$. The equilibrium solution is above or below the body 2.
- $\beta_1^{1/3} > \sqrt{1 + \beta_2^{2/3}}$. The equilibrium solution is located to the right of the body 2 (see Figure 10).

EDITOR: PLACE FIGURE 8 HERE.

EDITOR: PLACE FIGURE 9 HERE.

EDITOR: PLACE FIGURE 10 HERE.

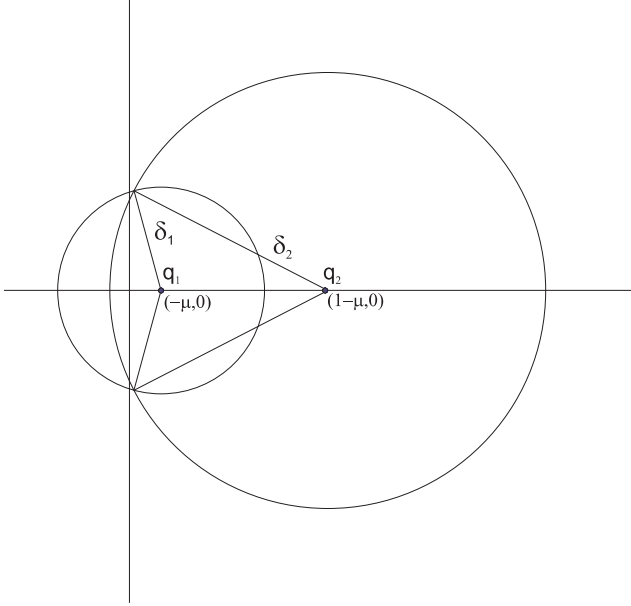


Fig. 8 Equilibrium solution situated to the left of the body 1.

On the other hand, for the collinear case, i.e. $y = 0$, the relative distances assume the form

$$\rho_1 = |x + \mu|, \quad \rho_2 = |x + \mu - 1|,$$

and the roots of the equation

$$F(x, \beta_1, \beta_2) = x f - \mu(1 - \mu) \left(\frac{\beta_1}{\rho_1^3} - \frac{\beta_2}{\rho_2^3} \right) = 0 \quad (18)$$

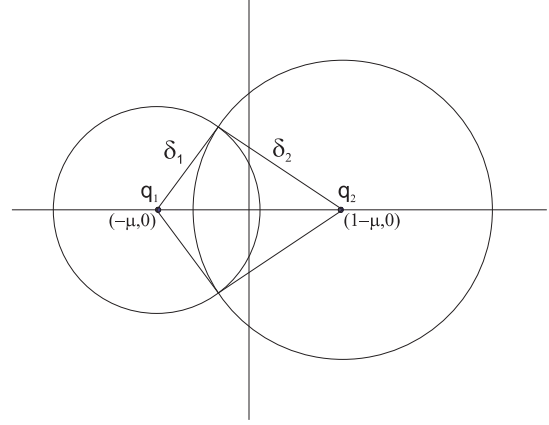


Fig. 9 Equilibrium solution situated between the bodies 1 and 2.

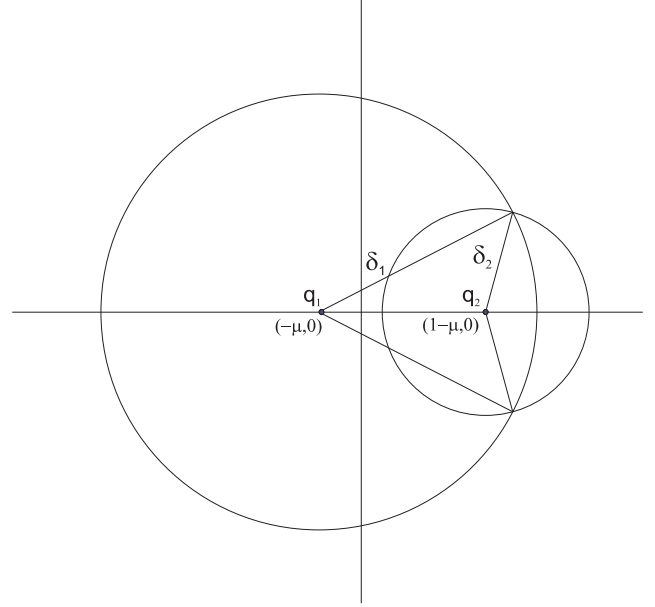


Fig. 10 Equilibrium solution situated to the right of the body 2.

define equilibrium solutions (by simplicity, we also use $F(x)$ to denote $F(x, \beta_1, \beta_2)$). The problem of the existence of the roots of F depends on β_1 and β_2 , and has been solved in parametric form by Lukyanov (1984), without the condition $(\beta_1 - 1)(\beta_2 - 1) < 1$. In the following we show necessary and sufficient conditions for the existence of the roots of F under such restriction. As first step, notice that (18) can be rewritten as

$$F(x) = x - \frac{\beta_1 (1 - \mu) (x + \mu)}{\rho_1^3} - \frac{\beta_2 \mu (x + \mu - 1)}{\rho_2^3}.$$

In order to facilitate the study of the function F , it is convenient to separate the domain in three intervals:

$$I_1 = (-\infty, -\mu), \quad I_2 = (-\mu, 1 - \mu), \quad I_3 = (1 - \mu, \infty).$$

We have the following reductions:

- $x \in I_1$. The relations $x + \mu = -\rho_1$, $x + \mu - 1 = -\rho_2$ are fulfilled, therefore $F(x)$ is reduced to $x + \frac{\beta_1(1-\mu)}{\rho_1^2} + \frac{\beta_2\mu}{\rho_2^2}$.
- $x \in I_2$. In this case $x + \mu = \rho_1$, $x + \mu - 1 = -\rho_2$, then $F(x)$ becomes $x - \frac{\beta_1(1-\mu)}{\rho_1^2} + \frac{\beta_2\mu}{\rho_2^2}$.
- $x \in I_3$. It is verified that $x + \mu = \rho_1$, $x + \mu - 1 = \rho_2$, therefore $F(x)$ assumes the form $x - \frac{\beta_1(1-\mu)}{\rho_1^2} - \frac{\beta_2\mu}{\rho_2^2}$.

Note that in the three intervals it is fulfilled

$$F'(x) = 1 + 2\beta_1(1-\mu)\frac{1}{\rho_1^3} + 2\beta_2\mu\frac{1}{\rho_2^3},$$

where the prime denotes differentiation respect to x . Next, we divide the plane (β_1, β_2) into four quadrants and two lines, that is

$$\begin{aligned} R_1 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 > 0, \beta_2 > 0\}, \\ R_2 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 < 0, \beta_2 > 0\}, \\ R_3 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 < 0, \beta_2 < 0\}, \\ R_4 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 > 0, \beta_2 < 0\}, \\ R_5 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 = 0\}, \\ R_6 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_2 = 0\}, \end{aligned}$$

and define the S -regions by means of

$$\begin{aligned} S_{1,1} &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid 0 < \beta_1 \leq 1, \beta_2 > 0\}, \\ S_{1,2} &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 > 1, 0 < \beta_2 < \frac{\beta_1}{\beta_1-1}\}, \\ S_2 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 < 0, \beta_2 > \frac{\beta_1}{\beta_1-1}\}, \\ S_{4,1} &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid 0 < \beta_1 < 1, \frac{\beta_1}{\beta_1-1} < \beta_2 < 0\}, \\ S_{4,2} &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 \geq 1, \beta_2 < 0\}, \\ S_5 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 = 0, \beta_2 > 0\}, \\ S_6 &= \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1 > 0, \beta_2 = 0\}. \end{aligned}$$

With this, we introduce the sets R'_i , $i = 1, \dots, 6$, which correspond respectively to R_i , $i = 1, \dots, 6$, restricted to $(\beta_1 - 1)(\beta_2 - 1) < 1$. Thus, we have

$$\begin{aligned} R'_1 &= S_{1,1} \cup S_{1,2}, \\ R'_2 &= S_2, \\ R'_3 &= \emptyset, \\ R'_4 &= S_{4,1} \cup S_{4,2}, \\ R'_5 &= S_5, \\ R'_6 &= S_6. \end{aligned}$$

These sets define the allowed regions for (β_1, β_2) . Depending on the region (with exception of R'_3), and the interval, could exist one or two collinear equilibrium solutions.

In the following we introduce a Theorem that relates different regions of the β -parameters for which the collinear equilibrium solutions exist. It shall be useful for demonstration of subsequent results.

Theorem 4.2 *Suppose that there exist collinear equilibrium solutions for (β_1, β_2) defined by the inequalities*

$$\begin{aligned} f_1(x_*, \mu) &\leq \beta_1 \leq f_2(x_*, \mu), \\ g_1(x_*, \mu) &\leq \beta_2 \leq g_2(x_*, \mu), \\ h_1(\mu) &\leq x_* \leq h_2(\mu), \end{aligned} \tag{19}$$

that is, for each pair (β_1, β_2) there exist x such that $F(x, \beta_1, \beta_2) = 0$. Then, there exist collinear equilibrium solutions for those (β_1, β_2) which satisfy

$$\begin{aligned} g_1(-x_*, 1-\mu) &\leq \beta_1 \leq g_2(-x_*, 1-\mu), \\ f_1(-x_*, 1-\mu) &\leq \beta_2 \leq f_2(-x_*, 1-\mu), \\ -h_2(1-\mu) &\leq x_* \leq -h_1(1-\mu). \end{aligned} \tag{20}$$

Proof:

We want to show that there exist some x for the new β -parameters such that this triad defines a root of F . Consider the change of variables $x = -\tilde{x}$, $x_* = -\tilde{x}_*$, $\mu = 1 - \tilde{\mu}$, $\beta_1 = \tilde{\beta}_2$, $\beta_2 = \tilde{\beta}_1$. With this, the inequalities in (19) take the form

$$\begin{aligned} f_1(-\tilde{x}_*, 1-\tilde{\mu}) &\leq \tilde{\beta}_2 \leq f_2(-\tilde{x}_*, 1-\tilde{\mu}), \\ g_1(-\tilde{x}_*, 1-\tilde{\mu}) &\leq \tilde{\beta}_1 \leq g_2(-\tilde{x}_*, 1-\tilde{\mu}), \\ h_1(1-\tilde{\mu}) &\leq -x_* \leq h_2(1-\tilde{\mu}). \end{aligned} \tag{21}$$

In the following, we also use the tilde notation for the variables related with \tilde{x} , \tilde{x}_* , $\tilde{\mu}$, $\tilde{\beta}_1$, $\tilde{\beta}_2$. Notice that the relative distances become $\rho_1^2 = \tilde{\rho}_2^2$, $\rho_2^2 = \tilde{\rho}_1^2$, therefore $F(x, \beta_1, \beta_2) = 0$ can be written as $F(\tilde{x}, -\tilde{\beta}_1, -\tilde{\beta}_2) = 0$. The later equation and (21) describe another set of β -parameters for which collinear equilibrium solutions exist. Removing the tilde in these relations we obtain (20).

Now we give two Theorems about the necessary and sufficient conditions for the existence of the collinear equilibrium solutions, in terms of the β -parameters and I_i , $i = 1, 2, 3$. The first Theorem deals with one simple root of F , whereas the second one is related to two roots. The regions of existence of the collinear equilibrium solutions are shown in Figures 11, 12 and 13.

Theorem 4.3

1. *Region R'_1 . There exists exactly one collinear equilibrium solution for $x \in I_1, I_2, I_3$.*
2. *Region R'_2 . There exists exactly one collinear equilibrium solution for $x \in I_3$.*
3. *Region R'_4 . There exists exactly one collinear equilibrium solution for $x \in I_1$.*

4. *Region R'_5 . There exists exactly one collinear equilibrium solution for $x \in I_3$. On the other hand, if $\beta_2 > 1$ holds, then there exists exactly one collinear equilibrium solution for $x \in I_1$. In a similar way, if $0 < \beta_2 < 1$ is satisfied, then there exists exactly one collinear equilibrium solution for $x \in I_2$.*
5. *Region R'_6 . There exists exactly one collinear equilibrium solution for $x \in I_1$. On the other hand, if $\beta_1 > 1$ holds, then there exists exactly one collinear equilibrium solution for $x \in I_3$. In a similar way, if $0 < \beta_1 < 1$ is satisfied, then there exists exactly one collinear equilibrium solution for $x \in I_2$.*

Proof:

According to Theorem 4.2, items 2 and 4 imply 3 and 5 respectively, therefore it is enough to demonstrate items 1, 2 and 4.

First item, interval I_1 . We notice that if $x \rightarrow -\infty$ then $F(x) \rightarrow -\infty$, and if $x \rightarrow -\mu$ then $F(x) \rightarrow \infty$, because $\beta_1 > 0$. It is clear that the function is strictly increasing since $F'(x) > 0$. The conclusion follows by the continuity of the function. A similar argument can be used to demonstrate the same item, intervals I_2 and I_3 , and fourth item, interval I_3 .

Fourth item, interval I_1 . In this case $F(x) = x + \frac{\beta_2 \mu}{\rho_2}$, and the function takes its maximum value at $x = -\mu$. Also notice that if $x \rightarrow -\infty$ then $F(x) \rightarrow -\infty$, and that $F(x)$ is strictly increasing. Therefore, the function could have only one root. For the existence of the root it is required $F(-\mu) > 0$, which implies $\beta_2 > 1$. A similar argument can be used to demonstrate the same item, interval I_2 .

Second item. We write the function $F(x)$ in terms of $\rho_2 = x + \mu - 1$, as a quotient of polynomials defined for $\rho_2 \in (0, \infty)$. The equilibrium solutions are defined by the positive roots of the fifth-degree polynomial in the numerator:

$$\begin{aligned} & \rho_2^5 + \rho_2^4(3 - \mu) + \rho_2^3(3 - 2\mu) + \\ & + \rho_2^2((1 - \mu)(1 - \beta_1) - \beta_2\mu) + \\ & - 2\beta_2\mu\rho_2 - \beta_2\mu = 0. \end{aligned}$$

In this polynomial, the coefficients of the quintic, quartic and cubic terms are positives, whereas the sign of the coefficient of the quadratic term depends on the values of β_1 and β_2 , and the remain coefficients are negative. This implies that there is only one variation in the sign of the sequence of the coefficients, independently of the coefficient of the quadratic term, therefore we have only one positive root.

In order to state the Theorem concerning two roots of F (possibly one root of multiplicity 2), we introduce a new parameter x_* ; it will be used to characterize the

frontier of the region of existence of the collinear solutions, on the parameters space (we proceed as Lukyanov (1984)). Once done that, we introduce

$$\begin{aligned} \beta_1^*(x_*, \mu) &= \frac{(3x_* + \mu - 1)(x_* + \mu)^3}{2(1 - \mu)}, \\ \beta_2^*(x_*, \mu) &= \frac{(3x_* + \mu)(x_* + \mu - 1)^3}{2\mu}, \end{aligned} \quad (22)$$

$$G(x_*, \mu) = \beta_1^*(x_*, \mu)\beta_2^*(x_*, \mu) - \beta_1^*(x_*, \mu) - \beta_2^*(x_*, \mu),$$

and $x_{r1}(\mu)$, $x_{r2}(\mu)$, to be defined in an implicit way. For fixed μ , the functions $x_{r1}(\mu)$, $x_{r2}(\mu)$ are specific roots of the eighth-degree polynomial $G(x_*, \mu)$ in x_* , for instance $G(x_{r1}(\mu), \mu) = 0$. The second root is defined, in terms of the first one, by the expression $x_{r2}(\mu) = -x_{r1}(1 - \mu)$. These functions satisfy $-\mu < x_{r1}(\mu) < -\frac{\mu}{3}$, $\frac{1}{3}(1 - \mu) < x_{r2}(\mu) < 1 - \mu$. At the Appendix we give an approximation of the roots $x_{r1}(\mu)$ and $x_{r2}(\mu)$, using regular perturbation theory.

The functions $x_{r1}(\mu)$ as the value of x_* where the curve $\beta_1 = \beta_1^*(x_*)$, $\beta_2 = \beta_2^*(x_*)$, $-\mu < x_* < -\frac{\mu}{3}$ meets $\beta_1\beta_2 - \beta_1 - \beta_2 = 0$, that is $G(x_{r1}(\mu), \mu) = 0$, we have $x_* \in (-\mu, x_{r1}(\mu))$ for the upper bound.

and $x_{r1}(\mu)$, $x_{r2}(\mu)$ as specific roots of the eighth-degree polynomial $G(x_*, \mu)$ in x_* . At the Appendix we give an approximation of the roots $x_{r1}(\mu)$ and $x_{r2}(\mu)$, using regular perturbation theory.

Theorem 4.4

1. *Region R'_2 . There are at most two collinear equilibrium solutions in each one of the intervals I_1 , I_2 . A necessary and sufficient condition for the existence of the equilibrium collinear solutions with $x \in I_1$ is $-\beta_1^*(x_*) \leq \beta_1 < 0$, $\beta_2^*(x_*) \leq \beta_2$, $x_* < -\mu$. On the other hand, the necessary and sufficient condition for $x \in I_2$ is $\beta_1^*(x_*) \leq \beta_1 < 0$, $\frac{\beta_1^*(x_*)}{\beta_1^*(x_*) - 1} < \beta_2 \leq \beta_2^*(x_*)$, $-\mu < x_* < x_{r1}(\mu)$.*
2. *Region R'_4 . There are at most two collinear equilibrium solutions in each one of the intervals I_3 , I_2 . A necessary and sufficient condition for the existence of the equilibrium collinear solutions with $x \in I_3$ is $-\beta_2^*(x_*) \leq \beta_2 < 0$, $\beta_1^*(x_*) \leq \beta_1$, $x_* > 1 - \mu$. On the other hand, the necessary and sufficient condition for $x \in I_2$ is $\beta_2^*(x_*) \leq \beta_2 < 0$, $\frac{\beta_2^*(x_*)}{\beta_2^*(x_*) - 1} < \beta_1 \leq \beta_1^*(x_*)$, $x_{r2}(\mu) < x_* < 1 - \mu$.*

Remark 4.2 *There are two types of collinear equilibrium solutions involved in Theorem 4.4. In one case the root of F is simple, whereas in the other one the root has multiplicity 2. The roots of multiplicity 2 only happen when the equalities for both β_1 and β_2 are fulfilled.*

For instance, $\beta_1 = -\beta_1^*(x_*)$, $\beta_2 = \beta_2^*(x_*)$, $x_* < -\mu$, in the first case of R'_2 .

Proof:

First item, interval I_1 . We write the function $F(x)$ in terms of $\rho_1 = -x - \mu$, as a quotient of polynomials, defined for $\rho_1 \in (0, \infty)$. The equilibrium solutions are defined by the positive roots of the polynomial of fifth degree in the numerator:

$$\begin{aligned} & -\rho_1^5 - (2 + \mu)\rho_1^4 - (1 + 2\mu)\rho_1^3 + \\ & + (\beta_1(1 - \mu) + (\beta_2 - 1)\mu)\rho_1^2 + \\ & + 2\beta_1(1 - \mu)\rho_1 + \beta_1(1 - \mu) = 0. \end{aligned}$$

Since $\beta_1 < 0$, possibly with exception of the coefficient of the quadratic term, the coefficients are negative, so it is possible to have two or none sign variations in the sequence of the coefficients, therefore we have three options: two simple roots, one root of multiplicity 2, or none root. This implies that $F(x)$ is a concave function, so there exists a local extrema x_* , that is $F'(x_*) = 0$. Notice that the condition $F(x_*) \geq 0$ guarantees the existence of at least one root. Thus, for a given μ , we want to determine those values of β_1 and β_2 such that both $F'(x_*) = 0$ and $F(x_*) \geq 0$ are fulfilled. With this aim, as first step we will introduce a change of variables $\beta_1, \beta_2 \rightarrow x_*, k$ such that x_* be a root of $F'(x)$ (in this case x_* defines a local minimum). Next, we will determine the allowed values for k ; to do that we use $F(x_*) \geq 0$ and $(\beta_1, \beta_2) \in R_2$. Finally, we will take into account the condition $(\beta_1 - 1)(\beta_2 - 1) < 1$ which gives rise to $(\beta_1, \beta_2) \in R'_2$. Let $x_* < -\mu$ the root of $F'(x)$. We introduce the change of variables

$$\begin{aligned} \beta_1 &= -\frac{1+k}{2(1-\mu)}(x_* + \mu)^3, \\ \beta_2 &= \frac{k}{2\mu}(1 - x_* - \mu)^3, \end{aligned}$$

which satisfies $F'(x_*) = 0$. The condition $F(x_*) \geq 0$ implies $k \geq -(3x_* + \mu)$, therefore β_1 and β_2 must be greater or equal to certain minimum values. Notice that $k = -(3x_* + \mu)$ defines the lower bound of β_1 and β_2 , that is $-\beta_1^*(x_*)$ and $\beta_2^*(x_*)$, respectively. Besides this, due to $(\beta_1, \beta_2) \in R_2$, we require $\beta_1 < 0$, so at this point the inequalities $-\beta_1^*(x_*) \leq \beta_1 < 0$, $\beta_2^*(x_*) \leq \beta_2$, $x_* < -\mu$ are satisfied. Since $x_* < -\mu$, we have $\beta_2 > 1$; using this and $\beta_1 < 0$ we conclude that the inequality $(\beta_1 - 1)(\beta_2 - 1) < 1$ holds, therefore $(\beta_1, \beta_2) \in R'_2$.

Second item, interval I_3 . It is consequence of the first item, interval I_1 , and Theorem 4.2

First item, interval I_2 . We proceed as for the demonstration of the first item, interval I_1 . We write the

function $F(x)$ in terms of $\rho = \frac{1}{x+\mu} - 1$, as a quotient of polynomials, defined for $\rho \in (0, \infty)$. We focus on the positive roots of fifth-degree polynomial in the numerator:

$$-\beta_1(1 - \mu)\rho^5 - 3\beta_1(1 - \mu)\rho^4 + ((\beta_2 - 1)\mu - 3\beta_1(1 - \mu))\rho^3 + ((1 - \beta_1)(1 - \mu) + 3\beta_2\mu)\rho^2 + 3\beta_2\mu\rho + \beta_2\mu = 0.$$

By hypothesis $\beta_1 < 0$ and $\beta_2 > 0$, therefore the coefficients present positive signs, possibly with exception of the coefficients of the quadratic and cubic terms. It is possible to have two or none sign variations in the sequence of the coefficients, therefore we have three options: two simple roots, one root of multiplicity 2, or none root, so $F(x)$ is a concave function. Let $x_* \in (-\mu, 1 - \mu)$ the root of $F'(x)$. We use the change of variables

$$\begin{aligned} \beta_1 &= -\frac{1+k}{2(1-\mu)}(x_* + \mu)^3, \\ \beta_2 &= \frac{k}{2\mu}(1 - x_* - \mu)^3, \end{aligned}$$

so $F'(x_*) = 0$. From $F(x_*) \leq 0$ we obtain $k \leq -(3x_* + \mu)$. Since $(\beta_1, \beta_2) \in R_2 \subset R'_2$ we require $k > 0$. Using both previous inequalities we obtain $x_* < -\frac{\mu}{3}$. Taking into account $F(x_*) \leq 0$, it is concluded that β_1 must be greater or equal to a minimum value, whereas β_2 must be lesser or equal to a maximum value, namely $\beta_1^*(x_*)$ and $\beta_2^*(x_*)$, respectively. Due to $(\beta_1, \beta_2) \in R_2$, we require $\beta_1 < 0$, $\beta_2 > 0$, therefore we have obtained $\beta_1^*(x_*) \leq \beta_1 < 0$, $0 < \beta_2 \leq \beta_2^*(x_*)$ for $-\mu < x_* < -\frac{\mu}{3}$. The final step is to consider $(\beta_1 - 1)(\beta_2 - 1) < 1$, or in an equivalent way $\beta_1\beta_2 - \beta_1 - \beta_2 < 0$. Notice that the β -parameters in which we are interested have the curve $(\beta_1^*(x_*), \beta_2^*(x_*))$, $-\mu < x_* < -\frac{\mu}{3}$, as upper bound, and $\beta_1\beta_2 - \beta_1 - \beta_2 = 0$ as lower bound. Defining $x_{r1}(\mu)$ as the value of x_* where the curve $\beta_1 = \beta_1^*(x_*)$, $\beta_2 = \beta_2^*(x_*)$, $-\mu < x_* < -\frac{\mu}{3}$, meets $\beta_1\beta_2 - \beta_1 - \beta_2 = 0$, that is $G(x_{r1}(\mu), \mu) = 0$, we have $x_* \in (-\mu, x_{r1}(\mu))$ for the upper bound. With the aim of obtaining a single inequality, for the region of existence, of the equilibrium solutions, we parameterize the lower bound, namely $\beta_1\beta_2 - \beta_1 - \beta_2 = 0$, using $x_* \in (-\mu, x_{r1}(\mu))$; we only must consider the part of the curve that goes from the point $(\beta_1^*(x_{r1}(\mu)), \beta_2^*(x_{r1}(\mu)))$ to the origin. The corresponding parameterization is $\beta_1 = \beta_1^*(x_*)$, $\beta_2 = \frac{\beta_1^*(x_*)}{\beta_1^*(x_*) - 1}$, $x_* \in (-\mu, x_{r1}(\mu))$. Therefore, the region of existence of the collinear equilibrium solutions

is defined by the inequalities

$$\begin{aligned}\beta_1^*(x_*) &\leq \beta_1 < 0, \\ \frac{\beta_1^*(x_*)}{\beta_1^*(x_*) - 1} &< \beta_2 \leq \beta_2^*(x_*), \\ -\mu &< x_* < x_{r1}(\mu).\end{aligned}$$

Second item, interval I_2 . It is consequence of the first item, interval I_2 , and Theorem 4.2. We identify $x_{r2}(\mu)$ with $-x_{r1}(1-\mu)$, therefore $-x_{r1}(1-\mu)$ must be a root of $G(x_*, \mu)$. According to (22), we have that $\beta_1^*(-x_*, 1-\mu) = \beta_2^*(x_*, \mu)$ and $\beta_1^*(-x_*, 1-\mu) = \beta_2^*(x_*, \mu)$, which implies $G(-x_*, 1-\mu) = G(x_*, \mu)$. From this we conclude $G(-x_{r1}(1-\mu), 1-\mu) = G(x_{r1}(\mu), \mu) = 0$, as required, for consistency.

EDITOR: PLACE FIGURE 11 HERE.

EDITOR: PLACE FIGURE 12 HERE.

EDITOR: PLACE FIGURE 13 HERE.

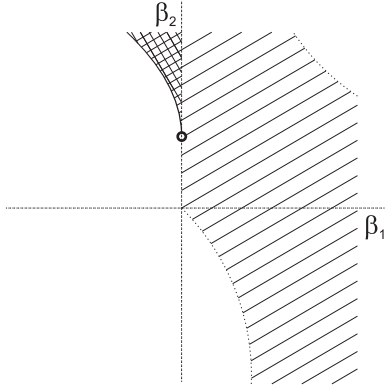


Fig. 11 Values of the β -parameters for which there exist collinear equilibrium solutions, on the interval I_1 . Each pair (β_1, β_2) belonging to the single hatched part allows only one equilibrium solution. On the other hand, the points associated to the double hatched part allow two different equilibrium solutions. Moreover, there exists only one equilibrium solution for the β -parameters belonging to the continuous border of the double hatched region; these parameters are associated to a root of F with multiplicity 2.

There exist collinear equilibrium solutions which have a very simple algebraic expression. These solutions are the limit of the triangular ones.

Theorem 4.5 Assuming that $(\beta_1, \beta_2) \in R'_1$, we have the following:

1. $(-\mu - \beta_1^{1/3}, 0)$ is a collinear equilibrium solution on the interval I_1 , whenever $\beta_2^{1/3} - \beta_1^{1/3} = 1$.
2. $(-\mu + \beta_1^{1/3}, 0)$ is a collinear equilibrium solution on the interval I_2 , whenever $\beta_1^{1/3} + \beta_2^{1/3} = 1$.

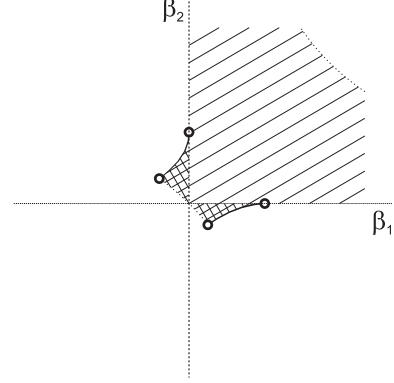


Fig. 12 Same description of Figure 11, on the interval I_2 .

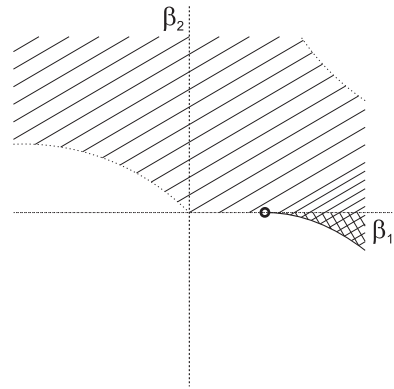


Fig. 13 Same description of Figure 11, on the interval I_3 .

3. $(-\mu + \beta_1^{1/3}, 0)$ is a collinear equilibrium solution on the interval I_3 , whenever $\beta_1^{1/3} - \beta_2^{1/3} = 1$.

Proof:

Due to the similarity of the proof of these items, we will prove only item 2. Since $x = -\mu + \beta_1^{1/3}$ we have $\rho_1 = \beta_1^{1/3}$. Notice that $\rho_2 = \beta_2^{1/3}$ holds, since $\beta_1^{1/3} + \beta_2^{1/3} = 1$. Thus, the condition on the relative distances $\rho_1 + \rho_2 = 1$ is true. Finally,

$$\begin{aligned}F(-\mu + \beta_1^{1/3}) &= -\mu + \beta_1^{1/3} - \beta_1^{1/3}(1-\mu) + \beta_2^{1/3}\mu \\ &= \mu(-1 + \beta_1^{1/3} + \beta_2^{1/3}) = 0.\end{aligned}$$

5 Stability of the equilibrium solutions

We know that the linear stability of the equilibrium solutions is determined by the eigenvalues of the matrix

$$A = J \text{ Hess } H(x^*, y^*, p_x^*, p_y^*),$$

where (x^*, y^*, p_x^*, p_y^*) is an equilibrium solution of the system (14) (see Meyer and Hall (1992)). Then,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ V_{xx} & V_{xy} & 0 & 1 \\ V_{xy} & V_{yy} & -1 & 0 \end{pmatrix}, \quad (23)$$

where

$$\begin{aligned} V_{xx} &= -\frac{\beta_1(1-\mu)}{\rho_1^3} - \frac{\beta_2\mu}{\rho_2^3} + \\ &+ 3\beta_1(1-\mu)\frac{(x+\mu)^2}{\rho_1^5} + 3\beta_2\mu\frac{(x+\mu-1)^2}{\rho_2^5}, \\ V_{xy} &= 3\beta_1(1-\mu)\frac{(x+\mu)y}{\rho_1^5} + 3\beta_2\mu\frac{(x+\mu-1)y}{\rho_2^5}, \\ V_{yy} &= -\frac{\beta_1(1-\mu)}{\rho_1^3} - \frac{\beta_2\mu}{\rho_2^3} + \\ &+ 3\beta_1(1-\mu)\frac{y^2}{\rho_1^5} + 3\beta_2\mu\frac{y^2}{\rho_2^5}. \end{aligned} \quad (24)$$

Using the relations $\rho_1 = \beta_1^{1/3}$, $\rho_2 = \beta_2^{1/3}$, which are true for triangular equilibrium solutions, or the collinear equilibrium solutions given by Theorem 4.5, (24) is reduced to

$$\begin{aligned} V_{xx} &= -1 + \frac{3(1-\mu)(x+\mu)^2}{\beta_1^{2/3}} + \frac{3\mu(x+\mu-1)^2}{\beta_2^{2/3}}, \\ V_{xy} &= \frac{3(1-\mu)(x+\mu)y}{\beta_1^{2/3}} + \frac{3\mu(x+\mu-1)y}{\beta_2^{2/3}}, \\ V_{yy} &= -1 + \frac{3(1-\mu)y^2}{\beta_1^{2/3}} + \frac{3\mu y^2}{\beta_2^{2/3}}. \end{aligned} \quad (25)$$

In order to study the characteristic polynomial associated to the matrix (23), in each equilibrium solution of (14), it is convenient to use $\Omega = \frac{1}{2}\mathbf{r}^2 + V$ (see equations (9) and (13)). The characteristic polynomial is

$$\lambda^4 + \lambda^2(4 - \Omega_{xx} - \Omega_{yy}) + \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = 0. \quad (26)$$

The triangular equilibrium solutions have been located using the sides of the triangle: ρ_1 , ρ_2 and 1. Now we use the parameterization considered by Lukyanov (1986), which consists of angles γ_i , $i = 1, 2$, and the unitary side; γ_i is the angle between the side ρ_i and 1, as it is shown in Figure 14.

EDITOR: PLACE FIGURE 14 HERE.

By means of using (25), the characteristic polynomial (26) becomes

$$\lambda^4 + \lambda^2 + 9\mu(1-\mu)\sin^2(\gamma) = 0, \quad (27)$$

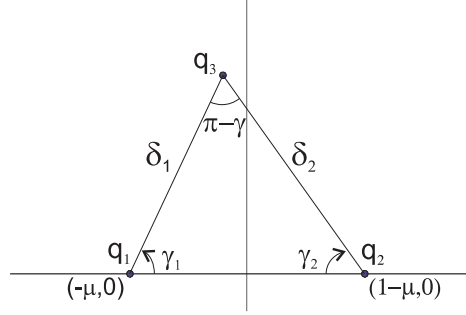


Fig. 14 Angles γ_1 and γ_2 .

where $\gamma = \gamma_1 + \gamma_2$, $\gamma \in [0, \pi]$. Notice that the characteristic equation of the collinear equilibrium solutions, described by Theorem 4.5, is obtained by setting $\gamma = 0, \pi$. It is straightforward to solve (27) using the change of variable $u = \lambda^2$. The eigenvalues of A are given by

$$\lambda = \pm \sqrt{\frac{-1 \pm \sqrt{F(\mu, \gamma)}}{2}},$$

where $F(\mu, \gamma) = 1 - 36\mu(1-\mu)\sin^2(\gamma)$. The curve $F = 0$ is defined by the equation $36\mu(1-\mu)\sin^2(\gamma) = 1$, with $\frac{1}{2} - \frac{\sqrt{2}}{3} \leq \mu \leq \frac{1}{2}$, $\frac{1}{9} \leq \sin^2(\gamma) \leq 1$. Since $\mu^* = \frac{1}{2} - \frac{\sqrt{2}}{3}$ is the smallest value of the mass parameter such that the function F can be zero, the associated angle must be $\gamma = \frac{\pi}{2}$. For $\mu \in (\mu^*, \frac{1}{2}]$, always there exist angles $\gamma = \gamma_\mu, \pi - \gamma_\mu$, where $\gamma_\mu = \arcsin(\frac{1}{6\sqrt{\mu(1-\mu)}})$, in a such way that $F(\mu, \gamma) = 0$ (γ_μ and $\pi - \gamma_\mu$ are different from each other). In particular, for $\mu = \frac{1}{2}$ the corresponding angles are γ_0 and $\pi - \gamma_0$, where $\gamma_0 = \arcsin(\frac{1}{3})$.

The stability of the triangular equilibrium solutions is determined by the value of F ; in the following Theorem we state four possible cases. Although the equilibrium solutions limit of the triangular ones (collinear equilibrium solutions discussed in Theorem 4.5) are not triangular, we have included them in the fourth item of following Theorem.

Theorem 5.1 *The triangular equilibrium solutions, as well as the solutions limit of the triangular ones, of the circular restricted charged three-body problem, satisfy:*

1. *For the values of (μ, γ) such that $F(\mu, \gamma) < 0$, the equilibrium solutions are unstable in the Lyapunov sense.*
2. *For the values of (μ, γ) such that $F(\mu, \gamma) = 0$, the equilibrium solutions are linearly unstable.*
3. *For the values (μ, γ) such that $0 < F(\mu, \gamma) < 1$, the equilibrium solutions are linearly stable.*
4. *For the values (μ, γ) such that $F(\mu, \gamma) = 1$, the equilibrium solutions are linearly unstable.*

Proof:

In order to cover all possible cases, observe that $-8 \leq F(\mu, \gamma) \leq 1$ holds for $\mu \in (0, \frac{1}{2}]$, $\gamma \in [0, \pi]$.

First item. In this case the eigenvalues of the matrix A are $\lambda = \pm \sqrt{\frac{-1 \pm i\sqrt{|F(\mu, \gamma)|}}{2}}$. The conclusion follows since the real part of the eigenvalues is different from zero.

Second item. We observe that the eigenvalues of the matrix A are $\lambda = \pm \frac{\sqrt{2}}{2} i$, each one with multiplicity 2. The matrix A is non diagonalizable because for each λ its eigenvector is

$$\left(\frac{2\lambda + V_{xy}}{D}, \frac{\lambda^2 - 1 - V_{xx}}{D}, \frac{\lambda^2 + 1 + V_{xx} + \lambda V_{xy}}{D}, 1 \right),$$

where $D = \lambda^3 + \lambda(1 - V_{xx}) + V_{xy}$. Thus, the equilibrium solutions are linearly unstable.

Third item. The eigenvalues of the matrix A are $\lambda = \pm i\sqrt{\frac{1 \pm \sqrt{F(\mu, \gamma)}}{2}}$. The solution is linearly stable, since all eigenvalues are pure imaginary and distinct.

Fourth item. In this case the eigenvalues of the matrix A are $\lambda = 0$ with multiplicity 2, and $\lambda = \pm i$, so A is non diagonalizable. The conclusion follows. Notice that $F(\mu, \gamma) = 1$ can only happen for $\gamma = 0, \pi$.

In Figure 15 are indicated the regions described by Theorem 5.1. Similar results were previously obtained by Lukyanov (1986).

EDITOR: PLACE FIGURE 15 HERE.

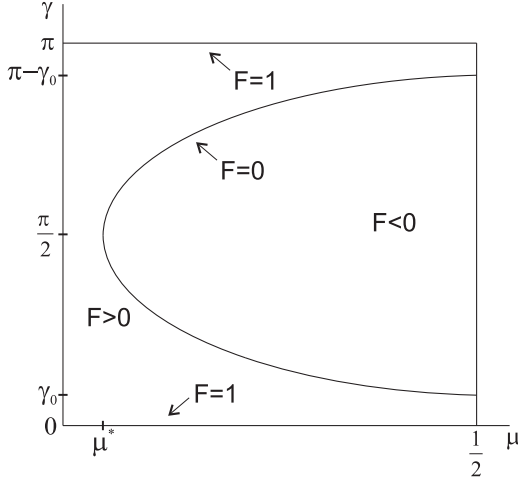


Fig. 15 Description of the values of F as function of (μ, γ) .

5.1 Stable region in the configuration space

In order to study the stable region in the configuration space, we define the restricted configuration space as the points (x, y) which meet $\rho_1 \neq 0$, $\rho_2 \neq 0$, $\rho_1 + 1 \geq \rho_2$, $\rho_2 + 1 \geq \rho_1$, $\rho_1 + \rho_2 \geq 1$, $(\rho_1^3 - 1)(\rho_2^3 - 1) < 1$. With

this, we avoid collisions. Moreover, all the triangular equilibrium solutions have physical sense, according to Theorem 4.1.

We want to show the evolution of the stable region, in the restricted configuration space, as μ increases. As first step, we apply the law of cosines to the triangle of sides 1, ρ_1 , ρ_2 , and angle of interest $\pi - \gamma$. From this we get

$$\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos(\gamma) = 1. \quad (28)$$

By geometry, we know that $\gamma = \text{constant}$ defines two arcs in the restricted configuration space, for fixed μ . One arc satisfies $y \geq 0$, whereas the other one $y \leq 0$; we will refer to them as upper and lower arcs, respectively. With the aim of describing the arcs, we write (28) in terms of (x, y) . After some algebraic manipulations, (28) becomes

$$\left(x - \frac{1}{2} + \mu \right)^2 + \left(y \pm \frac{\cos(\gamma)}{2\sin(\gamma)} \right)^2 = \frac{1}{4\sin^2(\gamma)}. \quad (29)$$

The upper(lower) arc is defined by the circumference (29) with plus(minus) sign.

According to Theorem 5.1, and Figure 15, if $\mu \in (0, \mu^*)$ then the stable region is characterized by $\gamma \in (0, \pi)$, and at $\mu = \mu^*$ appears the unstable region, associated to $\gamma = \frac{\pi}{2}$, so the corresponding stable region is defined by $\gamma \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. On the other hand, for fixed $\mu \in (\mu^*, \frac{1}{2}]$, the stable region is conformed by the points which satisfy $\gamma \in (0, \gamma_\mu) \cup (\pi - \gamma_\mu, \pi)$, where $\gamma_\mu = \arcsin(\frac{1}{6\sqrt{\mu(1-\mu)}})$. In the following, it is described the evolution of the stable region, in the restricted configuration space, according to the variation of the mass parameter.

- $0 < \mu < \mu^*$. Exception made of the points along the horizontal axis, all the triangular equilibrium solutions are stable, since $0 < F(\mu, \gamma) < 1$. It is outlined in Figure 16.

EDITOR: PLACE FIGURE 16 HERE.

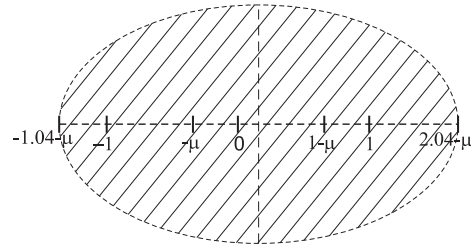


Fig. 16 Stable region (hatched part), for $\mu \in (0, \mu^*)$, in the restricted configuration space.

- $\mu = \mu^*$. The unstable region is conformed by the points on the horizontal axis, and those defined by $\gamma = \frac{\pi}{2}$. Except for the mentioned points, all the triangular equilibrium solutions are stable. It is shown in Figure 17.

EDITOR: PLACE FIGURE 17 HERE.

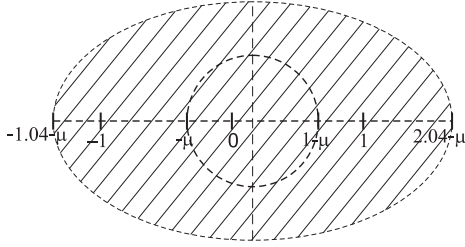


Fig. 17 Stable region (hatched part), for $\mu = \mu^*$, in the restricted configuration space. The dashed circumference indicates the unstable region associated to $\gamma = \frac{\pi}{2}$.

- $\mu^* < \mu \leq \frac{1}{2}$. The stable region, for fixed μ , is defined by the points which meet $\gamma \in (0, \gamma_\mu) \cup (\pi - \gamma_\mu, \pi)$, where $\gamma_\mu = \arcsin(\frac{1}{6\sqrt{\mu(1-\mu)}})$. It is outlined in Figure 18.

EDITOR: PLACE FIGURE 18 HERE.

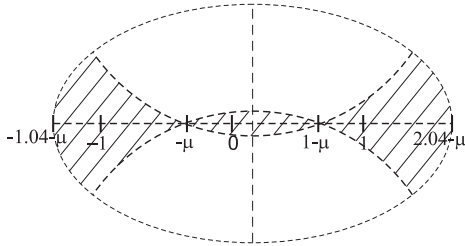


Fig. 18 Stable region (hatched part), for fixed $\mu \in (\mu^*, \frac{1}{2}]$, in the restricted configuration space. The dashed arcs are associated to $\gamma = \gamma_\mu, \pi - \gamma_\mu$.

5.2 Stable region in the parameters space

The evolution of the stable region also can be studied on the parameters space (δ_1, δ_2) . In analogy to what was done for the configuration space, we define the restricted parameters space as the points (δ_1, δ_2) which satisfy $\delta_1 \neq 0$, $\delta_2 \neq 0$, $\delta_1 + 1 \geq \delta_2$, $\delta_2 + 1 \geq \delta_1$, $\delta_1 + \delta_2 \geq 1$, $(\delta_1^3 - 1)(\delta_2^3 - 1) < 1$. Similar results were previously shown by Simmons (1985), without the restriction on β_1 and β_2 .

The equation $\delta_1^2 + \delta_2^2 + 2\cos(\gamma)\delta_1\delta_2 = 1$ (see (28)) defines an ellipse with center $(0, 0)$, and semi-axes

$\frac{1}{\sqrt{1 \pm \cos(\gamma)}}$, rotated $\pm \frac{\pi}{4}$ radians in a counterclockwise sense with respect to the horizontal axis (both signs hold due to the dependence, of the semi-axes, on the harmonic function). Notice that the ellipses defined by γ and $\pi - \gamma$ are related through a rotation of $\frac{\pi}{2}$ radians.

As we did for the restricted configuration space, we describe the evolution of the stable region, in the restricted parameters space, according to the variation of μ .

- $0 < \mu < \mu^*$. With the exception of the points that satisfy $\gamma = 0, \pi$, all the triangular equilibrium solutions are stable. It is outlined in Figure 19.

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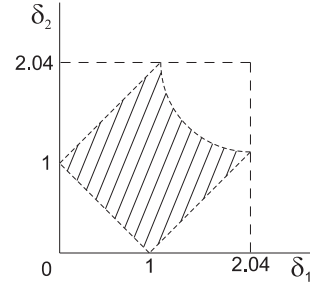


Fig. 19 Stable region (hatched part), for $\mu \in (0, \mu^*)$, in the restricted parameters space.

- $\mu = \mu^*$. The unstable region is conformed by the points which meet $\gamma = 0, \pi, \frac{\pi}{2}$. With the exception of these points, all the triangular equilibrium solutions are stable. It is outlined in Figure 20.

EDITOR: PLACE FIGURE 20 HERE.

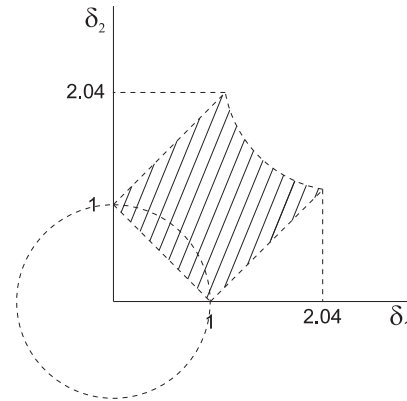


Fig. 20 Stable region (hatched part), for $\mu = \mu^*$, in the restricted parameters space. The dashed circumference indicates the unstable region associated to $\gamma = \frac{\pi}{2}$.

- $\mu^* < \mu \leq \frac{1}{2}$. The stable region, for fixed μ , is constituted by the points which satisfy $\gamma \in (0, \gamma_\mu) \cup (\pi - \gamma_\mu)$, where $\gamma_\mu = \arcsin(\frac{1}{6\sqrt{\mu(1-\mu)}})$. It is shown in Figure 21.

EDITOR: PLACE FIGURE 21 HERE.

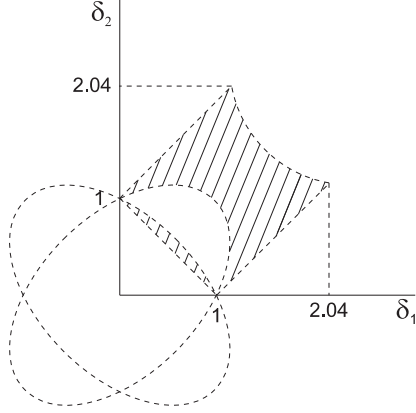


Fig. 21 Stable region (hatched part), for fixed $\mu \in (\mu^*, \frac{1}{2}]$, in the restricted configuration space. The curves $\gamma = \gamma_\mu, \pi - \gamma_\mu$ are denoted with dashed ellipses.

6 Conclusions

The condition $(\beta_1 - 1)(\beta_2 - 1) < 1$ must be fulfilled by the parameters β_1 and β_2 , for the proper establishment of the planar circular restricted charged three-body problem. As a consequence of such inequality, the triangular equilibrium solutions are confined to a certain region of the configuration space. In a similar way, such restriction reduces the possible values of β_1 and β_2 for which collinear equilibrium solutions exist.

As happens in the classical restricted case, the stability of the triangular solutions is guaranteed with a small mass parameter μ . It is interesting that, although the collinear equilibrium solutions (limit of the triangular ones) are linearly unstable, near of them appear stable equilibrium solutions, at least in the linear sense.

In this work, we have considered the triangular equilibrium solutions in the plane, as well as its linear stability, and the existence of the collinear equilibrium solutions. It would be interesting to study the stability of the equilibrium collinear solutions which are not limit of those triangular ones, as well as the equilibrium solutions in the space.

A Approximations of the roots $x_{r1}(\mu)$ and $x_{r2}(\mu)$

In order to characterize the roots x_{r1} , x_{r2} of $G(x_*, \mu)$, we use a regular perturbation approach, considering μ as parameter of perturbation. Instead of work with $G(x_*, \mu)$, which is not well defined for $\mu = 0$, we deal with $\tilde{G}(x_*, \mu) = 4\mu(1 - \mu)G(x_*, \mu)$, which is a polynomial in μ . We remember that such roots satisfy $-\mu < x_{r1} < -\frac{1}{3}\mu$, $\frac{1}{3}(1 - \mu) < x_{r2} < 1 - \mu$. Notice that in the limit $\mu \rightarrow 0$ we have $\tilde{G}(x_*, \mu) \rightarrow 3x_*(x_* - 1)^4(2 + x_* + 2x_*^2 + 3x_*^3)$, therefore $x_{r1} \rightarrow 0$ and $x_{r2} \rightarrow 1$. For x_{r1} we have a regular perturbation problem. We approximate x_{r1} in a power series (four terms), that is

$$x_{r1} = \sum_{i=1}^4 a_i \mu^i.$$

Equating to zero the coefficients of the powers of μ in $\tilde{G}(x_{r1}, \mu) = 0$, we get $a_1 = -\frac{1}{3}$, $a_2 = 0$, $a_3 = 0$, $a_4 = -\frac{8}{81}$, therefore the required approximation becomes

$$x_{r1} = -\frac{1}{3}\mu - \frac{8}{81}\mu^4.$$

One the other hand, x_{r2} cannot be handled directly by a regular perturbation approach. Nevertheless, by means of the change of variable $x_{r2} = 1 + \epsilon z$, where $\epsilon = \mu^{1/4}$, we obtain a regular problem with z as variable, and ϵ as perturbation parameter. Using

$$z = \sum_{i=0}^3 b_i \epsilon^i$$

in $\tilde{G}(1 + \epsilon z, \mu) = 0$ we get $b_0 = -\sqrt[4]{\frac{4}{27}}$, $b_1 = \frac{11}{36\sqrt{3}}$, $b_2 = \frac{67}{864\sqrt[4]{12}}$, $b_3 = \frac{497}{486}$, thus we obtain the approximation

$$x_{r2} = 1 - \sqrt[4]{\frac{4}{27}}\mu^{\frac{1}{4}} + \frac{11}{36\sqrt{3}}\mu^{\frac{1}{2}} + \frac{67}{864\sqrt[4]{12}}\mu^{\frac{3}{4}} - \frac{497}{486}\mu.$$

In this case, we had two possible values for b_0 , one positive, the other one negative. In order to satisfy the inequality $x_{r2} < 1 - \mu$, the negative vale was chosen.

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